

Lecture 26

Radiation Fields

We have shown how to connect the vector and scalar potentials to the sources \mathbf{J} and ρ of an electromagnetic system. This is a very important connection: it implies that once we know the sources, we know how to find the fields. But the relation between the fields and the sources are in general rather complex. In this lecture, we will simplify this relation by making a radiation field or far-field approximation by assuming that the point where the field is observed is very far from the source location in terms of wavelength. This approximation is very useful for understanding the physics of the radiation field from a source such as an antenna. It is also important for understanding the far field of an optical system. As shall be shown, this radiation field carries the energy generated by the sources to infinity.

26.1 Radiation Fields or Far-Field Approximation

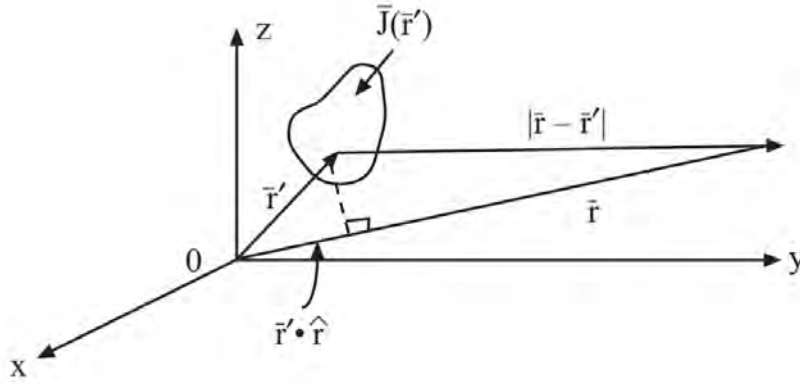


Figure 26.1: The relation of the observation point located a \mathbf{r} to the source location at \mathbf{r}' . The distance of the observation point \mathbf{r} to the source location \mathbf{r}' is $|\mathbf{r} - \mathbf{r}'|$.

In the previous lecture, we have derived the relation of the vector and scalar potentials to the sources \mathbf{J} and ρ .¹ They are given by

$$\mathbf{A}(\mathbf{r}) = \mu \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad (26.1.1)$$

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon} \iiint_V d\mathbf{r}' \rho(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad (26.1.2)$$

where $\beta = \omega\sqrt{\mu\epsilon} = \omega/c$ is the wavenumber. The integrals in (26.1.1) and (26.1.2) are normally untenable, but when the observation point is far from the source, approximation to the integrals can be made giving them a nice physical interpretation.

¹This topic is found in many standard textbooks in electromagnetics [31, 42, 49]. They are also found in lecture notes [39, 129].

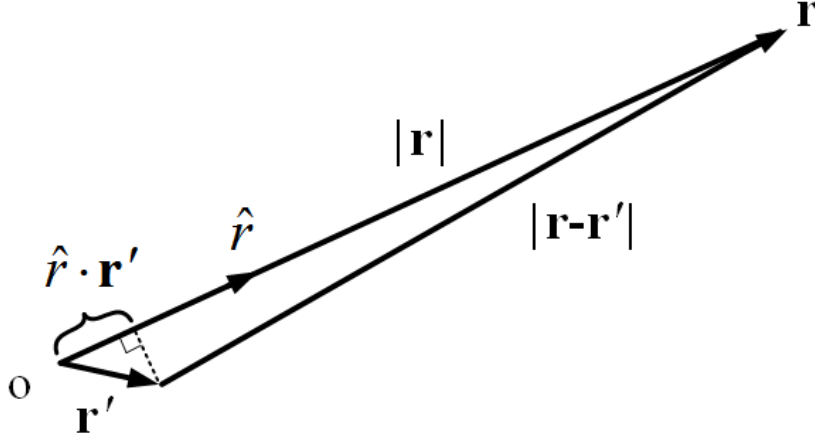


Figure 26.2: The relation between $|\mathbf{r}|$ and $|\mathbf{r} - \mathbf{r}'|$ using the parallax method, or that $|\mathbf{r} - \mathbf{r}'| \approx |\mathbf{r}| - \mathbf{r}' \cdot \hat{\mathbf{r}}$. It is assumed that \mathbf{r} is almost parallel to $\mathbf{r} - \mathbf{r}'$.

26.1.1 Far-Field Approximation

When $|\mathbf{r}| \gg |\mathbf{r}'|$, then $|\mathbf{r} - \mathbf{r}'| \approx r - \mathbf{r}' \cdot \hat{\mathbf{r}}$, where $r = |\mathbf{r}|$. This approximation can be shown algebraically or by geometrical argument as shown in Figure 26.2. Thus (26.1.1) above becomes

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu}{4\pi} \iiint_V d\mathbf{r}' \frac{\mu \mathbf{J}(\mathbf{r}')}{r - \mathbf{r}' \cdot \hat{\mathbf{r}}} e^{-j\beta r + j\beta \mathbf{r}' \cdot \hat{\mathbf{r}}} \approx \frac{\mu e^{-j\beta r}}{4\pi r} \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') e^{j\beta \mathbf{r}' \cdot \hat{\mathbf{r}}} \quad (26.1.3)$$

In the above we have made use of that $1/(1 - \Delta) \approx 1$ when Δ is small, but $e^{j\beta \Delta} \neq 1$, unless $j\beta \Delta \ll 1$. Hence, we keep the exponential term in (26.1.3) but simplify the denominator to arrive at the last expression above.

If we let $\boldsymbol{\beta} = \beta \hat{\mathbf{r}}$, and $\mathbf{r}' = \hat{x}x' + \hat{y}y' + \hat{z}z'$, then

$$e^{j\beta \mathbf{r}' \cdot \hat{\mathbf{r}}} = e^{j\boldsymbol{\beta} \cdot \mathbf{r}'} = e^{j\beta_x x' + j\beta_y y' + j\beta_z z'} \quad (26.1.4)$$

Therefore (26.1.3) resembles a 3D Fourier transform integral, namely

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu e^{-j\beta r}}{4\pi r} \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') e^{j\boldsymbol{\beta} \cdot \mathbf{r}'} \quad (26.1.5)$$

and (26.1.5) can be rewritten as

$$\mathbf{A}(\mathbf{r}) \cong \frac{\mu e^{-j\beta r}}{4\pi r} \mathbf{F}(\boldsymbol{\beta}) \quad (26.1.6)$$

where

$$\mathbf{F}(\boldsymbol{\beta}) = \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') e^{j\boldsymbol{\beta} \cdot \mathbf{r}'} \quad (26.1.7)$$

is the 3D Fourier transform of $\mathbf{J}(\mathbf{r}')$ with $\boldsymbol{\beta} = \hat{r}\beta$.

It is to be noted that this is not a normal 3D Fourier transform because $|\boldsymbol{\beta}|^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2$. In other words, the length of the vector $\boldsymbol{\beta}$ is fixed to be β , whereas in a normal 3D Fourier transform, β_x , β_y , and β_z are independent variables, each with values in the range $[-\infty, \infty]$, and $\beta_x^2 + \beta_y^2 + \beta_z^2$ ranges from zero to infinity.

The above is the 3D “Fourier transform” of the current source $\mathbf{J}(\mathbf{r}')$ with Fourier variables, β_x , β_y , β_z lying on a sphere of radius β and $\boldsymbol{\beta} = \beta\hat{r}$. This spherical surface in the Fourier space is also called the Ewald’s sphere.

26.1.2 Locally Plane Wave Approximation

We can write \hat{r} or $\boldsymbol{\beta}$ in terms of direction cosines in spherical coordinates or that

$$\hat{r} = \hat{x} \cos \phi \sin \theta + \hat{y} \sin \phi \sin \theta + \hat{z} \cos \theta \quad (26.1.8)$$

Hence,

$$\mathbf{F}(\boldsymbol{\beta}) = \mathbf{F}(\beta\hat{r}) = \mathbf{F}(\beta, \theta, \phi) \quad (26.1.9)$$

It is not truly a 3D function, since β , the length of the vector $\boldsymbol{\beta}$, is fixed. It is a 3D Fourier transform with data restricted on a spherical surface.

Also in (26.1.6), when $r \gg \mathbf{r}' \cdot \hat{r}$, $e^{-j\beta r}$ is now a rapidly varying function of r while, $\mathbf{F}(\boldsymbol{\beta})$ is only a slowly varying function of \hat{r} or of θ and ϕ , the observation angles. In other words, the prefactor in (26.1.6), $\exp(-j\beta r)/r$, can be thought of as resembling a spherical wave. Hence, if one follows a ray of this spherical wave and moves in the r direction, the predominant variation of the field is due to $e^{-j\beta r}$, whereas the direction of the vector $\boldsymbol{\beta}$ changes little, and hence, $\mathbf{F}(\boldsymbol{\beta})$ changes little. Furthermore, \mathbf{r}' in (26.1.7) are restricted to small or finite number, making $\mathbf{F}(\boldsymbol{\beta})$ a weak function of $\boldsymbol{\beta}$.

The above shows that in the far field, the wave radiated by a finite source resembles a spherical wave. Moreover, a spherical wave resembles a plane wave when one is sufficiently far from the source. Hence, we can write $e^{-j\beta r} = e^{-j\boldsymbol{\beta} \cdot \mathbf{r}}$ where $\boldsymbol{\beta} = \hat{r}\beta$ and $\mathbf{r} = \hat{r}r$ so that a spherical wave resembles a plane wave locally. This phenomenon is shown in Figure 26.3.

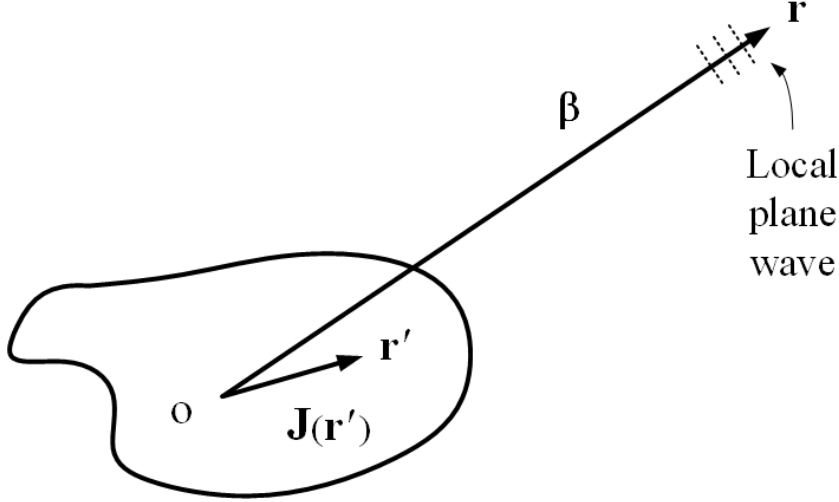


Figure 26.3: A source radiates a field that resembles a spherical wave. In the vicinity of the observation point \mathbf{r} , the field is strongly dependent on $\exp(-j\beta r)$ but weakly dependent on β . Hence, the field becomes locally a plane wave in the far field.

Then, it is clear that with the plane-wave approximation, $\nabla \rightarrow -j\beta = -j\beta\hat{r}$, we have

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \approx -j\frac{\beta}{\mu} \hat{r} \times (\hat{\theta}A_{\theta} + \hat{\phi}A_{\phi}) = j\frac{\beta}{\mu} (\hat{\theta}A_{\phi} - \hat{\phi}A_{\theta}) \quad (26.1.10)$$

Similarly [39, 129],

$$\mathbf{E} = \frac{1}{j\omega\epsilon} \nabla \times \mathbf{H} \cong -j\frac{\beta}{\omega\epsilon} \hat{r} \times \mathbf{H} \cong -j\omega(\hat{\theta}A_{\theta} + \hat{\phi}A_{\phi}) \quad (26.1.11)$$

Notice that $\beta = \beta\hat{r}$, the direction of propagation of the local plane wave, is orthogonal to \mathbf{E} and \mathbf{H} in the far field, a property of a plane wave. Moreover, there are more than one way to derive the electric field \mathbf{E} . For instance, using (26.1.10) for the magnetic field, the electric field can also be written as

$$\mathbf{E} = \frac{1}{j\omega\mu\epsilon} \nabla \times \nabla \times \mathbf{A} \quad (26.1.12)$$

Using the formula for the double-curl operator, the above can be rewritten as

$$\mathbf{E} = \frac{1}{j\omega\mu\epsilon} (\nabla\nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}) \cong \frac{1}{j\omega\mu\epsilon} (-\beta\beta + \beta^2\bar{\mathbf{I}}) \cdot \mathbf{A} \quad (26.1.13)$$

where we have used that $\nabla \cong -j\beta$ and $\nabla^2 \mathbf{A} = -\beta^2 \mathbf{A}$. Alternatively, we can factor $\beta^2 = \omega^2 \mu\epsilon$ and rewrite the above as

$$\mathbf{E} = -j\omega (-\hat{\beta}\hat{\beta} + \bar{\mathbf{I}}) \cdot \mathbf{A} = -j\omega (-\hat{r}\hat{r} + \bar{\mathbf{I}}) \cdot \mathbf{A} \quad (26.1.14)$$

Since $\bar{\mathbf{I}} = \hat{r}\hat{r} + \hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}$, then the above becomes

$$\mathbf{E} = -j\omega (\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}) \cdot \mathbf{A} = -j\omega(\hat{\theta}A_{\theta} + \hat{\phi}A_{\phi}) \quad (26.1.15)$$

which is the same as previously derived. It also shows that the electric field is transverse to the $\boldsymbol{\beta}$ vector. We can also arrive at the above by letting $\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi$, and using the appropriate formula for the scalar potential.

Furthermore, it can be shown that in the far field, using the plane-wave approximation,

$$|\mathbf{E}|/|\mathbf{H}| \approx \eta \quad (26.1.16)$$

where η is the intrinsic impedance of free space, which is a property of a plane wave. Moreover, one can show that the time average Poynting's vector, or the power density flow, in the far field is

$$\langle \mathbf{S} \rangle \approx \frac{1}{2\eta} |\mathbf{E}|^2 \hat{r} \quad (26.1.17)$$

which resembles also the property of a plane wave. Since the radiated field is a spherical wave, the Poynting's vector is radial. Therefore,

$$\langle \mathbf{S} \rangle = \hat{r}S_r(\theta, \phi), \quad \text{where} \quad S_r(\theta, \phi) = \frac{1}{2\eta} |\mathbf{E}|^2 \quad (26.1.18)$$

where S_r is the radial power density. The plot of $|\mathbf{E}(\theta, \phi)|$ is termed the far-field pattern or the radiation pattern of an antenna or the source, while the plot of $|\mathbf{E}(\theta, \phi)|^2$ is its far-field power pattern.

26.1.3 Directive Gain Pattern Revisited

Once the far-field radiation power pattern or the radial power density S_r is known, the total power radiated by the antenna can be found by integrating over all angles, viz.,

$$P_T = \int_0^\pi \int_0^{2\pi} r^2 \sin\theta d\theta d\phi S_r(\theta, \phi) \quad (26.1.19)$$

The above evaluates to a constant independent of r due to energy conservation. Now assume that this same antenna is radiating isotropically in all directions, then the average power density of this fictitious isotropic radiator as $r \rightarrow \infty$ is

$$S_{\text{av}} = \frac{P_T}{4\pi r^2} \quad (26.1.20)$$

A dimensionless directive gain pattern can be defined such that [31, 129]

$$G(\theta, \phi) = \frac{S_r(\theta, \phi)}{S_{\text{av}}} = \frac{4\pi r^2 S_r(\theta, \phi)}{P_T} \quad (26.1.21)$$

This directive gain pattern is a measure of the radiation power pattern of the antenna or source compared to when it radiates isotropically. The above function is independent of r in

the far field since $S_r \sim 1/r^2$ in the far field. As in the Hertzian dipole case, the directivity of an antenna $D = \max(G(\theta, \phi))$, is the maximum value of the directive gain. It is to be noted that by its mere definition,

$$\int d\Omega G(\theta, \phi) = 4\pi \quad (26.1.22)$$

where $\int d\Omega = \int_0^{2\pi} \int_0^\pi \sin\theta d\theta d\phi$. It is seen that since the directive gain pattern is normalized, when the radiation power is directed to the main lobe of the antenna, the corresponding side lobes and back lobes will be diminished.

An antenna also has an effective area or aperture A_e , such that if a plane wave carrying power density denoted by S_{inc} impinges on the antenna, then the power received by the antenna, P_{received} is given by

$$P_{\text{received}} = S_{\text{inc}} A_e \quad (26.1.23)$$

A wonderful relationship exists between the directive gain pattern $G(\theta, \phi)$ and the effective aperture, namely that²

$$A_e = \frac{\lambda^2}{4\pi} G(\theta, \phi) \quad (26.1.24)$$

Therefore, the effective aperture of an antenna is also direction dependent. The above implies that the radiation property of an antenna is related to its receiving property. This is a consequence of reciprocity theorem that we will study later. The constant of proportionality, $\lambda^2/(4\pi)$ is a universal constant that is valid for all antennas satisfying reciprocity theorem. The derivation of this constant for a Hertzian dipole is given in Kong [31], or using blackbody radiation law [129, 130].

The directivity and the effective aperture can be enhanced by designing antennas with different gain patterns. When the radiative power of the antenna can be directed to be in a certain direction, then the directive gain and the effective aperture (for that given direction) of the antenna is improved. This is shown in Figure 26.4. Such focussing of the radiation fields of the antenna can be achieved using reflector antennas or array antennas. Array antennas, as shall be shown, work by constructive and destructive wave field of the antenna.

Being able to do point-to-point communications at high data rate is an important modern application of antenna array. Figure 26.5 shows the gain pattern of a sophisticated antenna array design for 5G applications.

²The proof of this formula is beyond the scope of this lecture, but we will elaborate on it when we discuss reciprocity theorem.

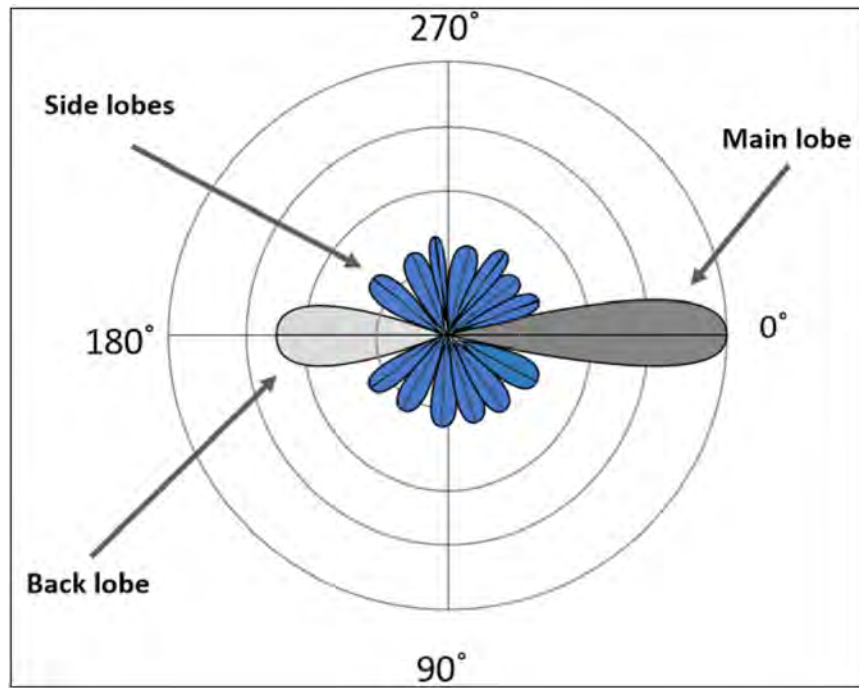


Figure 26.4: The directive gain pattern of an array antenna. The directivity is increased by constructive interference (courtesy of Wikipedia).

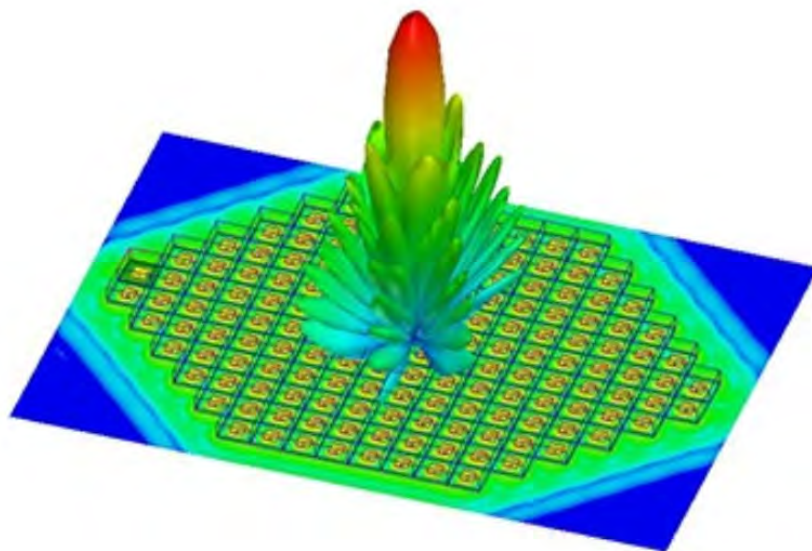


Figure 26.5: The directive gain pattern of a sophisticated array antenna for 5G applications (courtesy of Ozeninc.com).

